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Stochastic resonance enhanced by dichotomic noise in a bistable system

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We study linear responses of a stochastic bistable system driven by dichotomic noise to a weak periodic signal. We show that the effect of stochastic resonance can be greatly enhanced in comparison with the conventional case when dichotomic forcing is absent, that is, both the signal-to-noise ratio and the spectral power amplification reach much greater values than in the standard stochastic resonance setup.

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Stochastic resonance (SR) [1] has attracted much attention for the last two decades [2] because it is a generic phenomenon observed in a wide class of nonlinear systems simultaneously perturbed by a noise and a signal. SR is manifested by the existence of a certain nonzero noise intensity at which the ability of a system to transduce an information signal is maximized. Besides theoretical interests, SR has found applications in engineering [3] and indeed in biophysics, where it was demonstrated experimentally at the different levels of considerations, starting from ion channels [4] and single neurons [5,6] up to psychophysical experiments [7,8]. Finally, very recently SR has been demonstrated in behavioral experiments with the paddle fish [9].

A canonical model for SR is an overdamped doublewell potential system driven simultaneously by noise and a weak periodic signal. The spectral power amplification (SPA), that is, the weight of the signal part in the output spectrum, and the signal-to-noise ratio (SNR) serve as measures for SR. Both measures show the nonmonotonous behavior as functions of the noise intensity. For a practical application, however, the problem of enhancement of SR is of great importance. By the enhancement of SR we mean that SPA and/or SNR can reach larger values in comparison with standard stochastic bistable systems. It has been shown that SR can be enhanced if a bistable element embedded into the network of coupled bistable oscillators is taken instead of a single one [10,11]. SR can also be controlled in a sense of suppression or enhancement by the modulation of a system's threshold with correlated signals [12]. In this paper we propose a model of single stochastic bistable system driven additionally by a dichotomic noise that is uncorrelated to the signal. We show that the addition of such noise can greatly enhance SR and in contradiction to [12] the effect is observed within the linear-response limit. Note that SR in a bistable system driven by dichotomic noise *only* was studied in [13].

In the absence of signal, the model was described in detail in [14,15] and is a canonical symmetrical bistable system additionally driven by a weak dichotomic Markovian noise. Thus the model possesses two statistically independent noise sources: (i) ''thermal'' white noise, which is responsible for stochastic switching between metastable states, and (ii) dichotomic noise influencing the switching events between states of the system. Note that the magnitude of the dichotomic noise is always small, so that it cannot induce transitions between metastable states of the system by itself, and the presence of thermal noise is always necessary. It was shown that the addition of dichotomic noise leads to dramatic changes in the behavior of the system. In particular, dichotomic noise can synchronize the switching events, so that the mean switching rate of the system is locked and equal to the flipping rate of dichotomic noise in a *finite* range of thermal noise intensity [14]. This ability to manipulate the switching rate of the system by dichotomic noise opens a possibility to control stochastic resonance, if a weak periodic signal is added to the input of the system. As in [14], we assume that the stochastic bistable system possesses two symmetric states $\sigma(t)=\pm 1$, which represent the position of a particle in the right or the left well of a bistable potential with the barrier ΔU. The transition rate between two wells is given by $a = a_0 \exp[-(\Delta U/D)]$ [16]. Later on we use dimensionless noise, amplitudes, and the energy barrier. The energy barrier we set to the value $\Delta U=0.25$. We also scale time t by the factor a_0 so the rates are without dimension. Dichotomic noise $\lambda(t)=\pm 1$ with the magnitude B and the flipping rate $0<\gamma \ll a_0$ modifies the transition rate, so that a general expression for the modified rate reads [14,15]

$$W_0(\sigma,\lambda)=\exp\left(-\frac{\Delta U+\sigma\lambda B}{D}\right). \tag{1}$$

This four-state system with two states of the output in the presence of two states of the dichotomic noise is then described by the master equation,

$$\frac{d}{dt}p(\sigma\lambda)=-W_0(\sigma,\lambda)p(\sigma,\lambda)+W_0(-\sigma,\lambda)p(-\sigma,\lambda) + \gamma[p(\sigma,-\lambda)-p(\sigma,\lambda)]. \tag{2}$$

The mean switching frequency (MSF) of the output $\sigma(t)$ was derived in [15] and is given by

$$\langle\omega\rangle_{out}=\frac{\pi}{2}\left(a_1+a_2-\frac{(a_2-a_1)^2}{a_1+a_2+2\gamma}\right), \tag{3}$$

where the rates $a_{1,2}$ are

$$a_{1,2}=\exp[-(\Delta U\pm B)/D]. \tag{4}$$

The next step is to add a weak periodic signal $s(t) = A\cos(\Omega t+\theta)$, where θ is an initial phase, A is the ampli-

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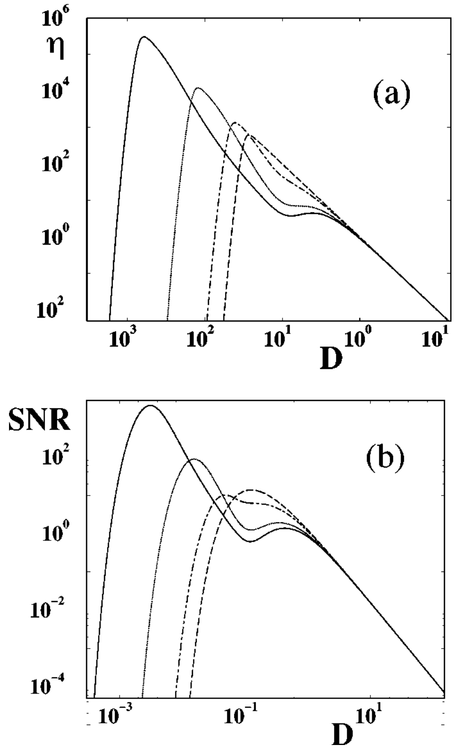

FIG. 1. SPA (a) and SNR (b) vs noise intensity D for different values of the magnitude of dichotomic noise B: $B=0$ (long-dashed line), $B=0.1$ (dash-dotted line), $B=0.2$ (dotted line), and $B=0.24$ (solid line). Other parameters are $\gamma=0.1$ and $\Omega=0.001$.

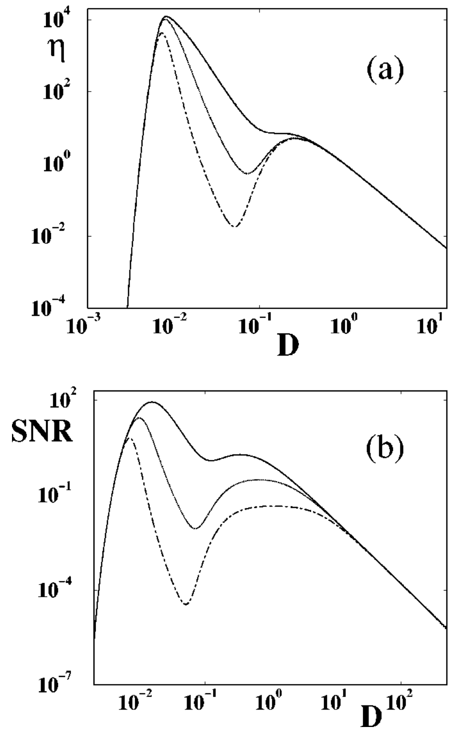

FIG. 2. SPA (a) and SNR (b) vs noise intensity D for different values of the flipping rate of dichotomic noise γ: 0.1 (solid line), 0.01 (dotted line), 0.001 (dash-dotted line). Other parameters are $B=0.2$, $\Omega=0.001$.

tude and Ω is the frequency of the signal. A sufficiently slow ($\Omega<\gamma\ll a_0$) and weak $A\ll\Delta U$ harmonic force leads to the following modified rates [17]:

$$W(\sigma,\lambda)=W_0(\sigma,\lambda)\exp\left[-\frac{A}{D}\sigma\cos(\Omega t+\theta)\right]$$
$$\approx W_0(\sigma,\lambda)\left[1-\frac{A}{D}\sigma\cos(\Omega t+\theta)\right]. \qquad (5)$$

The autocorrelation function can be obtained from the master equation (2) with the modified rates (5). We therefore describe the stochastic dynamics of the system in terms of linear response theory (LRT) [18,19] with respect to the signal $s(t)$. The equation for the autocorrelation function reads

$$\frac{d}{dt}\langle\sigma(t)\sigma(t')\rangle=-(a_1+a_2)\langle\sigma(t)\sigma(t')\rangle+(a_2-a_1)$$
$$\times\langle\lambda(t)\sigma(t')\rangle+[(a_1+a_2)\langle\sigma(t')\rangle$$
$$-(a_2-a_1)\langle\sigma(t)\lambda(t)\sigma(t')\rangle]\frac{A}{D}$$
$$\times\cos(\Omega t+\theta),\quad t\geqslant t'. \qquad (6)$$

The last expression contains the three-dimensional order cumulant $\langle\sigma(t)\lambda(t)\sigma(t')\rangle$, which is multiplied by the inputting signal. In the framework of the linear response theory, we therefore need only the linear part in A/D of this cumulant [19], which does not depend on t:

$$\langle\sigma(t)\lambda(t)\sigma(t')\rangle\approx\frac{a_2-a_1}{a_1+a_2+2\gamma}\langle\sigma(t')\rangle$$
$$=\langle\sigma\lambda\rangle_{stat}\langle\sigma(t')\rangle_{stat} \qquad (7)$$

with

$$\langle\sigma(t')\rangle=(A/D)[a_2+a_1-(a_2-a_1)^2/(a_1+a_2+2\gamma)]$$
$$\times\cos(\Omega t'+\theta+\psi).$$

The equation for the cross-correlation function reads

$$\frac{d}{dt}\langle\lambda(t)\sigma(t')\rangle=-2\gamma\langle\lambda(t)\sigma(t')\rangle. \qquad (8)$$

Equations (6) and (8) must be solved with the initial conditions $\langle\sigma(t)\sigma(t)\rangle=1$ and $\langle\sigma(t)\lambda(t)\rangle=\langle\sigma\lambda\rangle_{stat}$ [14]. It yields, after the Fourier transform and averaging over the initial phase θ, the power spectrum

$$S(\omega)=N(\omega)+A^2\pi\eta\delta(\omega-\Omega). \qquad (9)$$

Here $N(\omega)$ is the power spectrum of the background:

$$N(\omega)=4\frac{a_1+a_2}{(a_1+a_2)^2+\omega^2}\left(1+\frac{(a_2-a_1)^2}{4\gamma^2+\omega^2}\right)$$
$$-4\frac{(a_2-a_1)^2}{(a_1+a_2+2\gamma)(4\gamma^2+\omega^2)}, \qquad (10)$$

and η is the spectral power amplification:

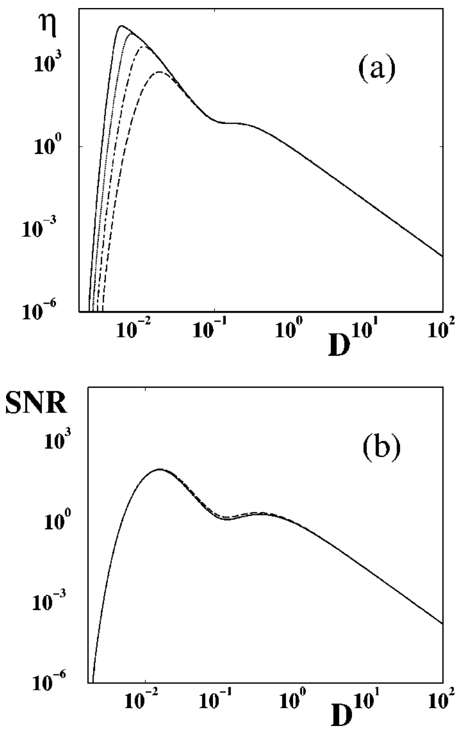

FIG. 3. SPA (a) and SNR (b) vs noise intensity D for different values of driving frequency Ω: 0.0001 (solid line), 0.001 (dotted line), 0.01 (dash-dotted line), and 0.1 (long-dashed line). Other parameters are $\gamma=0.1$, $B=0.2$.

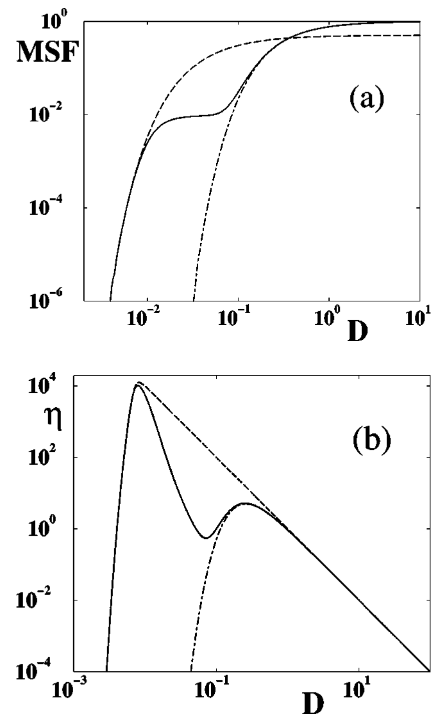

FIG. 4. MSF (a) and SPA (b) vs noise are shown by the solid lines for $B=0.2$, $\gamma=0.01$, $\Omega=0.001$. The limit of $\gamma\to\infty$ is shown by the dashed line and the limit of $\gamma\to 0$ is shown by the long-dashed line.

$$\eta=\frac{1}{D^2}\left[a_1+a_2-\frac{(a_2-a_1)^2}{(a_1+a_2+2\gamma)}\right]^2\frac{1}{(a_1+a_2)^2+\Omega^2}$$
$$=\frac{4}{\pi^2D^2}\frac{\langle\omega\rangle_{out}^2}{(a_1+a_2)^2+\Omega^2}, \tag{11}$$

where the MSF directly enters the equation for the SPA. The signal-to-noise ratio rescaled by the input amplitude in the LRT is defined as

$$R(\mathrm{SNR})=\pi\frac{\eta}{N(\Omega)}. \tag{12}$$

The SPA and the SNR are presented in Fig. 1 versus thermal noise intensity for the fixed values of the flipping rate γ and the signal frequency Ω but for different values of the magnitude of dichotomic noise. We immediately conclude that both the SPA and the SNR are greatly enhanced for a large enough B in comparison with the conventional case, when dichotomic noise is absent ($B=0$). Moreover, the behavior of the SPA and the SNR versus thermal noise intensity is qualitatively different from the conventional case, as both measures possess two maxima. Note that in contrast to the systems of coupled stochastic resonators [10,11], the optimal value of the noise intensity that maximizes SPA and SNR shifts towards smaller values with the increase of the magnitude of dichotomic noise (which plays a role of a coupling strength in our case).

In Fig. 2, SPA and SNR are presented for fixed values of B and Ω and for different values of the flipping rate of dichotomic noise γ. It can be seen that for large values of γ the second maximum of the SPA (large noise) disappears and only the first one (at small noise) remains. This first maximum is most pronounced for $\gamma>\Omega$. With the decrease of γ the first maximum occurs at noise intensities almost unchanged but decreases by its magnitude.

Finally, in Fig. 3 we show SPA and SNR versus D for different values of driving frequency. Although SNR almost does not depend on Ω for a given γ, the first peak of SPA depends strongly on Ω: both its magnitude and its position change. With the decrease of Ω, the magnitude of the first peak increases, while its position shifts towards smaller values of thermal noise intensity. In the limit of the fast switching, the location of this peak is determined by the matching condition: $\langle\omega\rangle_{out}\approx\Omega$, that is, the mean switching frequency of the unperturbed system (in the absence of a periodic signal) approximately equals the driving frequency. The second maximum of the SPA does not depend on the input frequency at all.

As the phenomenon of stochastic resonance is described above in terms of LRT, a physical interpretation of two-maxima behavior as well as the enhancement of SR can be done using discussion of the mean escape from a potential well [20,21] or, equivalently, the MSF. For this purpose two limiting cases are useful to consider: (i) $\gamma\to\infty$, that is, the limit of fast dichotomic noise. In this case the output mean frequency is determined by an effective potential with the lowered barrier $\Delta U-B$: $\langle\omega\rangle_{out}=\pi a_2/2=\pi/2\exp[-(\Delta U-B)/D]$. (ii) $\gamma\to 0$, that is, vanishingly slow dichotomic noise that can be treated as static asymmetry, $\langle\omega\rangle_{out}=\pi\exp(-\Delta U/D)/\cosh(B/D)$. The output mean switching frequency is presented in Fig. 4(a) along with the limiting cases (i) and (ii). For small noise intensity, the MSF approaches the case of $\gamma \rightarrow \infty$, while for large noise the MSF matches another limit of slow dichotomic noise $\gamma \rightarrow 0$. For intermediate values of D the MSF is locked and changes very slowly with the increase of noise intensity [14]. The corresponding behavior of the SPA is shown in Fig. 4(b). The small noise intensity region corresponds to a potential with a lowered barrier, so that SR may occur for a smaller noise intensity in comparison with the case of no dichotomic noise. That is why SR is greatly enhanced. This explains the first maximum in SPA and SNR dependence on D. In the intermediate range of noise intensity the mean escape rate is locked, that is, determined by dichotomic noise, and therefore the system is not sensitive to the external signal. Finally, for large noise intensity $\gamma \ll a_1, a_2$ the system behaves as an equivalent asymmetric one: during one round-trip switching of dichotomic noise the system performs many transitions between potential wells. It is known that SR is gradually suppressed with the increase of asymmetry of a bistable system and the position of the maxima in SNR or SPA does not depend on signal frequency [22]. This explains the second maximum in SPA and SNR dependence on D.

In conclusion, we studied SR in a bistable system subjected to both thermal noise and dichotomic noise. We have shown that SR can be controlled by dichotomic noise: the spectral power amplification and the signal-to-noise ratio can be greatly enhanced in comparison with the conventional case of no dichotomic noise.

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